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MONOMIAL CONDITIONS ON RINGS

BY

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ABSTRACT

Drazin introduced the notion of *pivotal monomial*, a condition on the evaluations of monomials in a ring, and characterized simple artinian rings as those primitive rings which have pivotal monomials. In this paper we consider monomial conditions related to pivotal monomials. The two major results are a characterization of prime Goldie rings in terms of pivotal monomials, and a characterization of the socle of a primitive ring in terms of generalized pivotal monomials.

1. Preliminaries

In this paper, all rings are associative, not necessarily with 1. Let R, R' be rings, with $1 \in R'$, such that R is an R'-bimodule. Define $Z\{X\}$ = free ring (without 1) generated by the countable set of noncommuting indeterminates $X_1, X_2, \cdots$; $Z\{X; t\}$ = subring of $Z\{X\}$ generated by $X_1, \cdots, X_t$. Let $\pi(t) =$ {monic monomials $h \in Z\{X\} \mid h \neq X_1 \cdots X_t$, and degree $h \geqq t$}, $\pi^k(t) = \pi(t) \cap Z\{X; k\}$. Say $y \in R'$ is *R-regular* if $yr \neq 0$ for all nonzero r in R; y is *strongly left R-regular* if $yr \neq 0$ and $ry \neq 0$ for all nonzero r in R, and if, given $b \neq 0$ in R, there are nonzero a_1, a_2 in R such that $a_1 y = a_2 b$ (i.e. Ry is left essential). Weakening Drazin's definition [4] of strong pivotal monomial, we say $X_1 \cdots X_t$ is *(R', R)-pivotal* (resp. *almost (R', R)-pivotal*) if, for each homomorphism $\varphi: Z\{X; t\} \to R$, one can find a strongly left R-regular (resp. R-regular) element y of R', such that $y\varphi(X_1 \cdots X_t) \in R'\varphi(\pi^t(t))$. Often R' will be the ring obtained by adjoining 1 formally to R; i.e. R' is the additive group $Z \oplus R$, endowed with multiplication $(n_1, r_1)(n_2, r_2) = (n_1 n_2, n_1 r_2 + n_2 r_1 + r_1 r_2)$, and the bimodule composition is given by $(n_1, r_1) r = n_1 r + r_1 r$ and $r(n_1, r_1) = r n_1 + r r_1$. In this case, (R', R)-pivotal (resp. almost pivotal) will merely be called R-pivotal (resp. almost pivotal). Clearly X_1 is almost R-pivotal for R a domain, since $\varphi(X_1)\varphi(X_1) \in \varphi(\pi^1(1))$. Drazin [4]

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observed that every simple artinian algebra satisfies pivotal monomials, and proved a converse with respect to *his* definitions (cf. Section 5). Pivotal monomials are defined in terms of subsets of R, rather than in terms of elements (as in the case of polynomial identities). One way of circumventing this difficulty is as follows:

Define $\pi(t,n) = \{h \in \pi'(t) \mid \text{degree } h \leqq n\}$, a finite set. (R',R) is (t,n)-*elementary* if, for each $r_1,\cdots,r_t$ in R, one can find $\{y_h \in R' \mid h \in \pi(t,n)\}$ and strongly left R-regular y in R', such that $yr_1\cdots r_t + \Sigma y_h h(r_1,\cdots,r_t) = 0$. If R' is the ring formed by adjoining 1 to R and if (R',R) is (t,n)-elementary, we shall say R is (t,n)-*elementary*.

PROPOSITION 1. *If R is an n-dimensional left vector space over a division ring D, then, for all $t, t' \geqq n$, R is $(t+1,\ t'+t+1)$-elementary; if, moreover, $1 \in R$, then R is $(n,2n)$-elementary.*

PROOF. For any r in R, clearly $\Sigma_{i=1}^{n+1} d_i r^i = 0$ for suitable d_i in D, not all 0. Multiplying by a suitable power of r, we have $r^{t+1} + \Sigma_{i=t+2}^{t+n+1} d'_i r^i = 0$ for suitable new d'_i in D, and the first assertion follows from a standard method of linearization (cf. [4]); the second assertion is analogous. Q.E.D.

In particular, $M_n(D)$, the $n \times n$ matrix ring over a division algebra D, is $(n^2, 2n^2)$-elementary. Clearly, if (R'_λ, R_λ) are (t,n)-elementary for all $\lambda \in \Lambda$, then $(\Pi R'_\lambda, \Pi R_\lambda)$ is (t,n)-elementary. If R is (t,n)-elementary for suitable t and n, we shall say R satisfies an *elementary condition*.

2. Prime rings with elementary condition

The object of this section is to characterize left orders in simple artinian rings in terms of elementary conditions. For completeness, we state the definitions and theorems which will be used, all of which are in Jacobson [8].

An element r of R is *regular* if $rr' \neq 0$ and $r'r \neq 0$ for all nonzero r' in R. A *classical left quotient ring* S of R is a ring containing R such that

(i) all elements of S have the form $r_1^{-1}r_2$, r_1 regular in R;

(ii) all regular elements of R are invertible in S.

R has a classical left quotient ring precisely when for each r_1, a_1 in R, r_1 regular, one can find r_2, a_2 in R, r_2 regular, such that $r_2 a_1 = a_2 r_1$. (This is Ore's condition.) If S is a classical left quotient ring of R, we say R is a *left order* in S.

Given subsets V, W of R, define $\text{Ann}_V(W) = \{r \in V \mid Wr = 0\}$ and $\text{Ann}'_V(W) = \{r \in V \mid rW = 0\}$. (If $V = R$ then we just write Ann W and Ann′ W.) A set $U = \text{Ann } W$ (resp. $= \text{Ann}' W$) is called a *right* (resp. *left*) *annihilator* and is a right (resp. left) ideal of R. U is *proper* if $0 \neq U \neq R$. For any left annihilator L, $\mathrm{Ann}'(\mathrm{Ann}\, L) = L$; for any right annihilator L', $\mathrm{Ann}(\mathrm{Ann}'\, L') = L'$. Hence, for left annihilators $L_1 \supset L_2$, we have $\mathrm{Ann}\, L_1 \subset \mathrm{Ann}\, L_2$. ($\supset$, $\subset$ will denote strict set containment.) Goldie has proved that R has a simple artinian (classical) left quotient ring precisely when:

(i) R is prime;

(ii) every strictly increasing chain of left annihilators in R is finite;

(iii) R does not contain an infinite direct sum of left ideals.

Such a ring R is called a *prime left Goldie ring*. We shall also need the

FAITH-UTUMI THEOREM. *Any order in $M_n(D)$, D a division ring, contains a subring of the form $M_n(T)$, T a domian with left quotient ring D.*

If R is a left order in S, then, for any $s_1, \cdots, s_m$ in S, one can find $r_1, \cdots, r_m, r$ in R, r regular, such that $s_i = r^{-1}r_i$, $1 \leqq i \leqq m$. Hence, the following result is an immediate consequence of the definitions:

PROPOSITION 2. *If R is a left order in a ring satisfying an elementary condition, then R satisfies an elementary condition.*

In particular, every prime left Goldie ring satisfies an elementary condition. We shall proceed to prove the converse, that every prime ring with elmentary condition is left Goldie. Half of this result is quite straightforward.

PROPOSITION 3. *Suppose R is prime and $X_1 \cdots X_t$ is almost (R', R)-pivotal. Then every chain of left annihilators in R has length at most $t+1$.*

PROOF. Suppose there is a chain of proper left annihilators $L_1 \subset L_2 \subset \cdots \subset L_t$, and let $T_i = \mathrm{Ann}\, L_i$, $1 \leqq i \leqq t$. Pick arbitrarily x_j in T_jL_j for all $j \leqq t$. Since $x_ix_j = 0$ for each $i \leqq j$, the only possible nonzero product of length $\geqq t$ of the x_i is $x_tx_{t-1} \cdots x_1$. Hence, by definition of almost pivotal monomial, $yx_t \cdots x_1 = 0$ for some R-regular y. Thus $x_t \cdots x_1 = 0$, so $0 = T_t(L_tT_{t-1})(L_{t-1}T_{t-2}) \cdots (L_2T_1)L_1$. But each L_iT_{i-1} is a nonzero ideal of R, contrary to the fact R is prime, so there cannot be a chain, of length t, of proper left annihilators. Since the only improper annihilators are R and 0, the assertion follows. Q.E.D.

NOTE. All chains of left annihilators of a ring have length $\leqq t+1$, if and only if all chains of right annihilators have length $\leqq t+1$. Indeed, suppose $T_1 \subset T_2 \subset \cdots \subset T_{t+2}$ is a chain of right annihilators. Then $\mathrm{Ann}'\, T_1 \supset \mathrm{Ann}'\, T_2 \supset \cdots \supset \mathrm{Ann}'\, T_{t+2}$ is a chain of left annihilators, proving $(\Rightarrow)$; $(\Leftarrow)$ is shown analogously.

To proceed further, we require some easy facts about annihilators.

PROPOSITION 4. *Assume all chains of left annihilators in* R *have length* $\leqq n$, *and let* $L_1 \subset \cdots \subset L_n$ *be a chain of left annihilators in* R.

(i) *Any chain of left (resp. right) annihilators in the ring* L_i *has length* $\leqq i$.

(ii) *If* R *is semiprime then* $L_1 \subset \cdots L_i$ *is a chain of left annihilators in* L_i *of maximal length.*

PROOF.

(i) Suppose $A_1 \subset \cdots \subset A_{i+1}$ is a chain of left annihilators in L_i. Then $\mathrm{Ann}\, A_1 \supset \cdots \supset \mathrm{Ann}\, A_{i+1} \supseteq \mathrm{Ann}\, L_i \supset \cdots \supset \mathrm{Ann}\, L_n$ are right annihilators in R, producing a chain of length $n+1$ of right annihilators of R, contrary to hypothesis. The assertion for right annihilators follows.

(ii) In view of (i), we need only show that each L_j is a left annihilator in L_i, for $j \leqq i$. Let $L = L_i$, $T_j = \mathrm{Ann}\, L_j$, $A_j = \mathrm{Ann}_L\, L_j = L \cap T_j$, $A'_j = \mathrm{Ann}'\, A_j$. Now $L_j \subseteq A'_j \cap L$. Moreover, $((A'_j \cap L)T_j)^2 \subseteq (A'_j \cap L)(T_jL)T_j \subseteq (A'_j \cap L)A_jT_j = 0$, so $A'_j \cap L \subseteq \mathrm{Ann}'T_j = L_j$ (since R is semiprime). Thus $L_j = A'_j \cap L = \mathrm{Ann}'_L A_j$, proving L_j is a left annihilator in L. Q.E.D.

PROPOSITION 5. *If* R *is prime and* $X_1 \cdots X_t$ *is* R*-pivotal, then* R *does* **not** *contain an infinite direct sum of left ideals.*

PROOF. By Proposition 3, there exists a chain of left annihilators $L_0 \subset \cdots \subset L_{n+1}$ of maximal length (with $L_0 = 0$ and $L_{n+1} = R$). Let $T_i = \mathrm{Ann}\, L_i$, $0 \leqq i \leqq n+1$. We are done unless L_{n+1} contains an infinite direct sum of left ideals of R. Thus, we assume inductively that L_{i+1} contains an infinite direct sum of left ideals of R and we claim that L_i contains an infinite direct sum of left ideals of R.

Let $L = L_{i+1}$, and let $B = \bigoplus_k B_k \subseteq L$ be an infinite direct sum of left ideals of R. We search for a nonzero element of $L_i \cap B$. For each k, note that $T_iB_k \not\subseteq T_{i+1}$ (for otherwise $LT_iB_k = 0$, implying $LT_i = 0$, contrary to L_i being a left annihilator); hence we can pick nonzero x_k in $T_iB_k - T_{i+1}$. Clearly $L \supset \mathrm{Ann}'_L\, x_k \supseteq L_i$. But, by Proposition 4(ii), L_i is a maximal annihilator in L, so $L_i = \mathrm{Ann}'_L\, x_k$. Define $\varphi : Z\{X; t\} \to R$ via $\varphi(X_k) = x_k$, all k. By definition of pivotal monomial, there exists strongly left R-regular y in R' such that $yx_1 \cdots x_t \in R'\varphi(\pi'(t))$. But comparing components of B yields $yx_1 \cdots x_t \in R'\varphi(\pi'(t-1))x_t$, i.e. $(yx_1 \cdots x_{t-1} - r)x_t = 0$ for some r in $R'\varphi(\pi'(t-1))$. Hence $(yx_1 \cdots x_{t-1} - r) \in B \cap \mathrm{Ann}\, x_t = B \cap L_i$, so our search is done unless $yx_1 \cdots x_{t-1} = r$, i.e. $yx_1 \cdots x_{t-1} \in R'\varphi(\pi'(t-1))$. Continuing in this way, we obtain a nonzero element of $L_i \cap B$ unless $yx_1 \in R'\varphi(\pi'(0))x_1$, i.e. $yx_1 = dx_1$ for some d in B. Since B is a direct sum, $d \in B_1 \oplus \cdots \oplus B_m$ for some m. Choose nonzero b in B_{m+1}. By definition of y, we can choose nonzero a_1, a_2 in R such that $a_1y = a_2b$.

Then $a_1dx_1 = a_2bx_1$, so $a_1d - a_2b \in B \cap \mathrm{Ann}\, x_1 = B \cap L_i$. Thus our search is done unless $a_1d = a_2b$. Matching components in B, we get $a_2b = 0$, implying $a_1y = 0$, contrary to y strongly left regular.

Thus we have a nonzero element b_1 of $L_i \cap B$. But $b_1 \in B_1 \oplus \cdots \oplus B_m$ for some m; letting $B' = \oplus\{B_k \mid k > m\}$, we apply the same argument to obtain nonzero b_2 in $L_i \cap B'$. Continuing this process gives us $b_1, b_2, \cdots$, and $\oplus\{Rb_j \mid 1 \leqq j < \infty\}$ is an infinite direct sum of nonzero left ideals of R, contained in L_i. This establishes the claim; applying the claim repeatedly gives an infinite direct sum of nonzero left ideals contained in $L_0 = 0$, which is ridiculous. Hence R cannot have contained an infinite direct sum of left ideals. Q.E.D.

Putting the various propositions together yields;

THEOREM 6. *The following are equivalent for a prime ring R:*

(i) *$X_1 \cdots X_t$ is R-pivotal.*

(ii) *R is a left order in $M_n(D)$, for suitable division ring D and some $n \leqq t$.*

(iii) *R is (t, m)-elementary for some m.*

PROOF.

(i)$\Rightarrow$(ii) By Propositions 3 and 5, R is left Goldie, so R is a left order in $M_n(D)$, for a suitable division ring D. Hence, by the Faith-Utumi Theorem, R contains a subring $M_n(T)$. Letting $\{e_{ij} \mid 1 \leqq i, j \leqq n\}$ be a set of matrix units, choose some x in T, and let $r_0 = 0$, $r_i = e_{11}x + \cdots + e_{ii}x$ for $i > 0$. Clearly $\mathrm{Ann}'\, r_0 \supset \mathrm{Ann}'\, r_1 \supset \cdots \supset \mathrm{Ann}'\, r_n$ in R, so, by Proposition 3, $n + 1 \leqq t + 1$. Hence $n \leqq t$.

(ii)$\Rightarrow$(iii) Immediate, by Proposition 2.

(iii)$\Rightarrow$(i) By definition. Q.E.D.

Theorem 6 is a characterization of prime Goldie rings, in terms of elementary conditions, so the object of this section has been achieved. Theorem 6 can be generalized to semiprime rings, using a technique due to Herstein, in his proof of Goldie's theorem for semiprime rings (cf. [7, pp. 174–176] or [8, appendix B]).

THEOREM 7. *The following are equivalent for a semiprime ring R:*

(i) *R satisfies the ascending chain condition on annihilators of 2-sided ideals, and $X_1 \cdots X_t$ is R-pivotal.*

(ii) *R is a left order in a finite direct sum of matrix rings of degree $\leqq t$ over division rings (in particular, R is semisimple artinian).*

(iii) *R satisfies the ascending chain condition on annihilators of 2-sided ideals, and R is (t, m)-elementary for some m.*

The only nontrivial implication in Theorem 7 is (i) $\Rightarrow$ (ii); in this case R can be viewed as a subdirect product of the minimal annihilators (of 2-sided ideals), each of which is a prime ring. The proof then parallels closely Herstein's proof cited above, and will be omitted.

3. Semiprime rings with almost pivotal monomial

The purpose of this section is to extend some of the structure theory of semiprime rings with polynomial identity to the theory of semiprime rings with almost pivotal monomial. (Note that this theory includes noncommutative domains.)

In this section, R is a semiprime ring, and R' is the ring with 1 formally adjoined to R.

LEMMA 8. *Suppose $X_1 \cdots X_t$ is almost R-pivotal and V is a subset of R. Then* $\mathrm{Ann}' V^j = \mathrm{Ann}' V^{j+1}$ *and* $\mathrm{Ann}\, V^j = \mathrm{Ann}\, V^{j+1}$ *for all $j \geqq t$.*

PROOF. Clearly it suffices to prove the lemma for $j = t$. Let $A_i = \mathrm{Ann}' V^i$, all i. For any $k > i$, $A_k V^{k-i} \subseteq A_i$, so $A_k V^{k-1} \subseteq A_i V^{i-1}$. Pick x_i in $V^i A_i$, each i. Since $x_i x_k = 0$ for $i \leqq k$, the only possible nonzero product, of length $\geqq t$, of the x_i is $x_t x_{t-1} \cdots x_1$. By definition of almost pivotal monomial, $y x_t x_{t-1} \cdots x_1 = 0$ for some R-regular y; hence $x_t \cdots x_1 = 0$. Thus, $0 = (V^t A_t) \cdots (V A_1) = V^t (A_t V^{t-1}) \cdots (A_2 V) A_1$, implying $(A_{t+1} V^t)^{t+1} = 0$. But R is semiprime, so $A_{t+1} \subseteq \mathrm{Ann}' V^t = A_t$. Hence $A_{t+1} = A_t$. One proves analogously that $\mathrm{Ann}\, V^t = \mathrm{Ann}\, V^{t+1}$. Q.E.D.

In particular, if V is nilpotent then, in the notation of Lemma 8, $\mathrm{Ann}\, V^t = \mathrm{Ann}\, V^{t+1} = \mathrm{Ann}\, V^{t+2} = \cdots = R$, proving $V^t = 0$. Recall that a left (resp. right) ideal is *essential* if it intersects nontrivially all left (resp. right) ideals. The *left singular ideal* $Z({}_R R) \equiv \{r \in R \mid \mathrm{Ann}' r$ is (left) essential$\}$, easily seen to be an ideal of R; the *right singular ideal* $Z({}_R R) \equiv \{r \in R \mid \mathrm{Ann}\, r$ is essential$\}$. Using a trick of Amitsur (cf. [8, ch. X, sec. 8]), we shall now obtain generalizations of theorems of Amitsur and of Fisher [5].

THEOREM 9. *Suppose $X_1 \cdots X_t$ is almost R-pivotal.*

(i) *Every nil subring of R is nilpotent of degree $\leqq t$.*

(ii) $Z({}_R R) = Z(R_R) = 0$.

PROOF.

(i) Let B be a nil subring of R. Since each element of B is nilpotent, Lemma 8 implies $b^t = 0$ for all b in B. Hence, by a theorem of Levitzki, B is locally nilpotent. Hence, for any $b_1, \cdots, b_t$ in B, $\{b_1, \cdots, b_t\}$ is nilpotent, implying $b_1 \cdots b_t = 0$, by Lemma 8. Therefore $B^t = 0$, proving (i).

(ii) Let $Z = Z({}_R R)$. We claim $x^t = 0$ for all x in Z. Indeed, if $x^t \neq 0$, we can choose nonzero b in $Rx^t \cap \mathrm{Ann}^l x$. Let $b = rx^t$. Then $r \in \mathrm{Ann}^l x^{t+1} = \mathrm{Ann}^l x^t$ (by Lemma 8), implying $0 = rx^t = b$, a contradiction. This establishes the claim; hence Z is a nil ideal of R. By (i), Z is nilpotent, implying $Z = 0$. The proof $Z(R_R) = 0$ is analogous. Q.E.D.

Theorem 9 (ii) implies that any semiprime ring with almost pivotal monomial has a maximal left quotient ring and a maximal right quotient ring, although these clearly need not be the same, as evidenced in particular by the existence of left Goldie domains which are not right Goldie (cf. [6]).

4. Restricted pivotal monomials and generalized pivotal monomials

DesMarrais [3] has introduced the notion of a restricted pivotal monomial, which can be put into the framework of this paper as follows: Let $\pi_1(t) =$ {multilinear monomials in $\pi(t,t)$}; $X_1 \cdots X_t$ is *restricted* (R', R)-pivotal if, for each homomorphism $\varphi: Z\{X\} \to R$, $\varphi(X_1 \cdots X_t) \in R'\varphi(\pi_1(t))$. This notion is very strong, per se, perhaps too strong as it stands. In fact, I believe that it is an open question whether or not all simple artinian rings satisfy restricted pivotal monomials. However, restricted pivotal monomials generalize very usefully, giving us a way to characterize the socle of a primitive ring.

Let S be a ring and $S\{X\}$ denote the free product of S and $Z\{X\}$. Note that each element of $S\{X\}$ is the (not necessarily unique) sum of elements of the form $r_1 X_{i_1} r_2 X_{i_2} \cdots X_{i_t} r_{t+1}$, r_i in S, X_i noncommuting indeterminates, $t \geqq 1$; such an element is called a *monomial with fingerprint* $X_{i_1} \cdots X_{i_t}$, and the r_i will be called the *coefficients*. A *generalized monomial* of $S\{X\}$ is an element of $S\{X\}$ which can be written as a sum of monomials with all the same fingerprints; it is not hard to see that every element of $S\{X\}$ can be written uniquely as a sum of generalized monomials. A generalized monomial of $S\{X\}$ is *multilinear* if its fingerprint is multilinear.

It this section we assume, as in Section 1, that R is an R'-bimodule, with the added condition that all possible associativity relationships hold between R and R': *homomorphisms* from $R'\{X\}$ to R will mean ring homomorphisms preserving the R'-bimodule structure and the associativity conditions. All homomorphisms from $R'\{X\}$ to R are determined by the action on the X_i; conversely, given $r_1, r_2, \cdots$ in R, there is a unique homomorphism $\varphi: R'\{X\} \to R$ such that $\varphi(X_i) = r_i$, all i.

Let $\pi_1(t; S, W) = \{$generalized monomials of $S\{X\}$ with fingerprints in $\pi_1(t)$ and coefficients in a *finite* subset W of $S\}$. $X_1 \cdots X_t$ is *generalized* (R', R)-*pivotal* if there exists (finite) $W \subset R'$ and a generalized monomial h in $R'\{X\}$ with fingerprint $X_1 \cdots X_t$ such that, for each homomorphism $\varphi: R'\{X\} \to R$, $\varphi(h) \in R'\varphi(\pi_1(t; R', W))$. (Compare with Amitsur [2] and DesMarrais [3].) Call h a *generalized pivotal monomial* of (R', R) and call each $\varphi(h)$ an *evaluation of* h. Note that we have in fact generalized the notion of *restricted* pivotal monomials (where $W = \{1\}$) for reasons that will begin to come clear in the subsequent paragraphs.

Consider the following situation through Theorem 11, using Jacobson [8, ch. II] as a general reference on primitive rings. R is left primitive; i.e. R has a faithful irreducible left module M. Let $D = \mathrm{End}_R M$, a division ring, and let R' be the subring of $\mathrm{End}\, M_D$ generated by 1 and R (which is dense in $\mathrm{End}\, M_D$). Viewing R' and D naturally in $\mathrm{End}_Z M$, let $S = R'D$. Note that $rd = dr$ for all r in R', d in D, and M is an S-module, under the action $(\Sigma_i r_i d_i)z = \Sigma_i (r_i z) d_i$, r_i in R', d_i in D, z in M.

Define $\mathrm{soc}\, R$ to be 0 unless R has nonzero minimal left ideals, in which case $\mathrm{soc}\, R$ is the sum of all nonzero minimal left ideals. If R is left primitive and L is a minimal left ideal of R, then L contains an idempotent element e and eRe is a division ring. Hence X_1 is restricted eRe-pivotal, implying eX_1e is a generalized pivotal monomial of R. Note that all evaluations of eX_1e clearly lie in $\mathrm{soc}\, R$; the main result of this section is that all evaluations of all generalized pivotal monomials of R lie in $\mathrm{soc}\, R$.

Note the canonical injections $R' :\hookrightarrow S$ and $D :\hookrightarrow S$ induce homomorphisms $\psi_1: R'\{X\} \to S\{X\}$ and $\psi_2: D\{X\} \to S\{X\}$. Let $R'\{X\}D$ denote the additive subgroup of $S\{X\}$ generated by elements of the form $\psi_1 f(X_1, \cdots, X_m)\psi_2 d$, f in $R'\{X\}$, d in D. For ease of notation, we shall merely write a typical element of $R'\{X\}D$ as $\Sigma f_i d_i$, f_i in $R'\{X\}$, d_i in D. Note that all generalized monomials and multilinearizations of elements of $R'\{X\}D$ are still in $R'\{X\}D$.

The reason behind the above machinations is that we cannot perform the usual "splitting" of a primitive ring and still be sure (a priori) of preserving its generalized pivotal monomials; hence we must always work in the context of M as a vector space over D. Let *subspace* denote finite-dimensional D-subspace of M. A generalized monomial $h(X_1, \cdots, X_m)$ of $R'\{X\}D$ is $(V, (u_i))$-*dominated* if there exist subspaces $V_1, \cdots, V_m$ of respective dimensions $u_1, \cdots, u_m$ and a finite set $W \subset S$ such that, for every homomorphism $\varphi: S\{X\} \to S$ with $\varphi(X_i) \in R$ and $\varphi(X_i)V_i = 0$, $1 \leqq i \leqq m$, and given z in M, we can find r in $R'\varphi(\pi_1(m; S, W))$ such that $\varphi(h)z - rz \in V$. We shall call W the *coefficient set of* h and the V_i will be called the *associated subspaces* (to $(V,(u_i))$). If h_1 and h_2 have fingerprint $X_1\cdots X_m$ and are $(V,(u_i))$-dominated with the same associated subspaces, then clearly h_1+h_2 is $(V,(u_i))$-dominated.

Assume a generalized monomial h of $R'\{X\}D$ has fingerprint $X_1\cdots X_m$ for suitable m. We can write $h=\Sigma_{j=1}^{v}h_j(X_1,\cdots,X_{m-1})X_ms_j$, suitable h_j in $R'\{X\}$, s_j in S, with v *minimal*, and define $\mathrm{ht}(h)$ inductively by $\mathrm{ht}(s)=1$ for all s in S and $\mathrm{ht}(h)=$ the smallest possible value of $m\Sigma_j ht(h_j)$, h written as above.

THEOREM 10. *Suppose V is a subspace of dimension u_0 and $h(X_1,\cdots,X_m)$ is a generalized monomial of $R'\{X\}D$, with fingerprint $X_1\cdots X_m$, which is $(V,(u_i))$-dominated for some $u_1,\cdots,u_m$. Let $\gamma=\mathrm{ht}(h)$, $u=\max(u_0,u_1,\cdots,u_m)$. There is a function $\tau_u(\gamma)$ such that, for each $x_1,\cdots,x_m$ in R,* $\mathrm{rank}\,h(x_1,\cdots,x_m)\leqq\tau_u(\gamma)$.

PROOF. Let $V_1,\cdots,V_m$ be the associated subspaces (to $(V,(u_i))$), and let W be the coefficient set of h. Write $h=\Sigma_{j=1}^{v}h_j(X_1,\cdots,X_{m-1})X_ms_j$ such that $m\Sigma\mathrm{ht}(h_j)=\gamma$. We may assume $s_jM\subseteq V_m$, $t<j\leqq v$, t minimal within this context. Also, expanding W if necessary, we may assume that each $s_j\in W$, $1\leqq j\leqq v$, and we list the elements of W as $\{s_k\mid 1\leqq k\leqq n\}$, where $n=\mathrm{card}\,W$. Now the proof will follow that of [11, theor. 1]. Define $\tau_u(0)=0$, all u, and define inductively $\tau_u(\gamma)=\max(u\gamma,2\tau_{u+n}(\gamma-1),\tau_{u'}(\gamma')$, all $u'<u$, all $\gamma'\leqq\gamma)$. If $\gamma=1$ then Theorem 10 is immediate; we work inductively on γ.

Let $h'=\Sigma_{j=1}^{t}h_jX_ms_j$, $h''=h-h'$. Clearly h'' is $(0,(u_i))$-dominated, so h' is $(V,(u_i))$-dominated, and $\mathrm{ht}(h')+\mathrm{ht}(h'')=\gamma$. If $h'\neq0$ and $h''\neq0$, then $\mathrm{ht}\,h'<\gamma$ and $\mathrm{ht}\,h''<\gamma$; then, for any $x_1,\cdots,x_m$ in R, $\mathrm{rank}\,h(x_1,\cdots,x_m)\leqq\tau_u(\mathrm{ht}\,h')+\tau_u(\mathrm{ht}\,h'')\leqq2\tau_u(\gamma-1)\leqq\tau_u(\gamma)$. If $h''=h$ then $\mathrm{rank}\,h(x_1,\cdots,x_m)\leqq u\gamma\leqq\tau_u(\gamma)$, all x_i. So we are done unless $h''=0$, i.e. $t=v$.

Hence $s_1z\notin V_m$ for some z in M. Now choose z' arbitrarily in M. By density, there exists x_m in R, $d_1,\cdots,d_n$ in D with $d_1=1$, such that $x_mV_m=0$ and $x_ms_kz=z'd_k$, all t, $1\leqq k\leqq n$. Moreover, the d_k are independent of the choice of z'. Let $h_1'=\Sigma_{j=1}^{v}h_jd_j$, $V_i'=V_i+\Sigma_ks_kzD$, $1\leqq i\leqq m-1$, and let $u_i'=\dim V_i'\leqq u_i+n$. We claim $h_1'(X_1,\cdots,X_{m-1})$ is $(V,(u_i'))$-dominated, with associated subspaces V_i'. Indeed, suppose we are given $x_1,\cdots,x_{m-1}$ in R such that $x_iV_i'=0$. Define $\varphi:S\{X\}\to S$ via $\varphi(X_i)=x_i$, $1\leqq i\leqq m-1$, $\varphi(X_m)=x_m$, and $\varphi(X_i)=0$ for $i>m$. Since h is $(V,(u_i))$-dominated, there exists r in $R'\varphi(\pi_1(m;S,W))$ such that $h(x_1,\cdots,x_m)z-rz\in V$. Now, by choice of the x_i, rz has the form $\Sigma_kr_kx_ms_kz$, r_k in $R'\varphi(\pi_1(m-1;S,W))$ and s_k in W. Then $rz=\Sigma r_kx_ms_kz=\Sigma r_kz'd_k=(\Sigma r_kd_k)z'$. Let $r'=\Sigma r_kd_k$ and let $W'=\{s_kd_j\mid 1\leqq k,j\leqq n\}$, a finite set of order n^2. Then $r'\in R'\varphi(\pi_1(m-1;S,W'))$ and $h_1'(x_1,\cdots,x_{m-1})z'-r'z'=$

$h(x_1, \cdots, x_m)z - rz \in V$. So, given z' in M we have found r' in $R'\varphi(\pi_1(m-1; S, W'))$ such that $h'_1(x_1, \cdots, x_{m-1})z' - r'z' \in V$, proving the claim.

For $m = 1$, $\mathrm{ht}(h'_1) = 1$; for $m > 1$, $\mathrm{ht}(h'_1) \leqq \gamma/m$. Thus $\mathrm{ht}(h'_1) \leqq \gamma - 1$. Therefore, for all $x_1, \cdots, x_{m-1}$ in R, $\mathrm{rank}\, h'_1(x_1, \cdots, x_{m-1}) \leqq \tau_{u+n}(\gamma - 1)$. Now let $\tilde{h} = \Sigma_{j=2}^{v} h_j X_m (s_j - d_j s_1)$, which has height $\leqq \gamma - m$. Now, for any x_m in R, $h'_1 x_m s_1$ is $(V, (u'_i))$-dominated. On the other hand, for all $x_1, \cdots, x_m$ in R,

$$\tilde{h}(x_1, \cdots, x_m) = \Sigma_{j=1}^{v} h_j(x_1, \cdots, x_{m-1}) x_m s_j - \Sigma_{j=1}^{v} h_j(x_1, \cdots, x_{m-1}) d_j x_m s_1$$

$$= h(x_1, \cdots, x_m) - h'_1(x_1, \cdots, x_{m-1}) x_m s_1 .$$

Hence, setting $u'_m = u_m$, we see that $\tilde{h}$ is $(V, (u'_i))$-dominated. Thus for all $x_1, \cdots, x_m$ in R, $\mathrm{rank}\, \tilde{h}(x_1, \cdots, x_m) \leqq \tau_{u+n}(\gamma - m)$, implying $\mathrm{rank}\, h(x_1, \cdots, x_m) \leqq \mathrm{rank}\, \tilde{h}(x_1, \cdots, x_m) + \mathrm{rank}\, h'_1(x_1, \cdots, x_{m-1}) \leqq 2\tau_{u+n}(\gamma - 1) \leqq \tau_u(\gamma)$, proving the theorem. Q.E.D.

Now every generalized pivotal monomial is $(0, (0))$-dominated, so we get immediately

THEOREM 11. *Every evaluation of a generalized pivotal monomial of a primitive ring lies in the socle.*

A generalized monomial h is *R-proper* if h is *not* a generalized identity of R (cf. [10, 11]). Theorem 11 contains the following result which is closely related to Amitsur [2]:

COROLLARY 12. *If a primitive ring R has a proper generalized pivotal monomial, then* $\mathrm{soc}\, R \neq 0$.

An interesting aspect of Theorem 11 is that it defines the socle of a primitive ring to be the set of evaluations of generalized pivotal monomials. This suggests that we define the *upper socle* $B(R)$ to be the set of evaluations of generalized pivotal monomials of R. Two questions which come to mind if R is prime are:

(1) If $B(R) \neq 0$, is the central closure of R (cf. [9]) a primitive ring with socle?

(2) If $1 \in B(R)$, is R left Goldie?

We shall proceed in a slightly different direction. Define the *index* of a simple artinian ring $M_n(D)$ to be n.

THEOREM 13. *Let R be semiprimitive with $1 \in R$, and let $\{R_\gamma \mid \gamma \in \Gamma\}$ be a set of (left) primitive homomorphic images of R. If R satisfies a generalized pivotal monomial which is proper for each nonzero homomorphic image R_γ, then $\{R_\gamma \mid \gamma \in \Gamma\}$ is a collection of simple artinian rings of bounded index.*

PROOF. Let h be a generalized pivotal monomial of R which is proper for every nonzero homomorphic image R_γ. Given r in R, let r_γ denote the canonical homomorphic image of r in R_γ. Each R_γ has a faithful irreducible left module M_γ with centralizer D_γ; viewing $R_\gamma \subseteq \mathrm{End}(M_\gamma)_{D_\gamma}$, let $A = \{r \in R \mid \{\mathrm{rank}\, r_\gamma \mid \gamma \in \Gamma\}$ is bounded$\}$, an ideal of R. If $A \neq R$ then $A \subseteq P$ for some maximal ideal P of R. But h is R/P-proper, contrary to Theorem 10. Thus $A = R$, so $1 \in A$, i.e. $\{\mathrm{rank}\, 1_\gamma \mid \gamma \in \Gamma\}$ is bounded. Thus each R_γ is simple artinian, with index $\leqq \max(\mathrm{rank}\, 1_\gamma \mid \gamma \in \Gamma)$. Q.E.D.

5. Comparison of definitions

In the notation of Section 1, say $X_1 \cdots X_t$ is *absolutely* (R', R)-*pivotal* if, for each homomorphism $\varphi: Z\{X; t\} \to R$, $\varphi(X_1 \cdots X_t) \in R'\varphi(\pi'(t))$. This definition is equivalent to Drazin's definition of what *he* calls strongly pivotal monomials, so, in particular, a primitive ring R is simple artinian of index $\leqq t$ if and only if $X_1 \cdots X_t$ is absolutely R-pivotal (cf. [4]). The obvious question that arises in correlating this paper to [4] is, "Do prime rings with pivotal monomial satisfy absolutely pivotal monomials?" We shall see shortly that the answer is negative, but first let us show that existence of almost pivotal, pivotal, or absolutely pivotal monomials is equivalent on primitive rings with socle:

LEMMA 14. *If R is a primitive ring with* $\mathrm{soc}\, R \neq 0$ *and if $X_1 \cdots X_t$ is almost R-pivotal, then R is simple artinian of index $\leqq t$.*

PROOF. Indeed, if the conclusion does not hold, then R contains a subring T isomorphic to $M_{t+1}(D)$, for a suitable division ring D. Let $\{e_{ij} \mid 1 \leqq i, j \leqq t+1\}$ be a suitable set of matric units of T, and define $\varphi: \mathbf{Z}\{X; t\} \to R$ via $\varphi(X_i) = e_{i,i+1}$. By definition of almost pivotal monomial (and since $e_{i,i+1}e_{j,j+1} = 0$ for $j \neq i+1$), we see that $ye_{12}e_{23} \cdots e_{t,t+1} = 0$ for some R-regular y in R'. But then $0 \neq ye_{1,t+1} = ye_{12} \cdots e_{t,t+1}$, a contradiction. Hence R must be simple artinian of index $\leqq t$. Q.E.D.

Now consider the following very well-known example: Let B be the ideal of $\mathbf{Q}\{X; 2\}$ generated by $X_1X_2 - X_2X_1 - 1$, and let $R = \mathbf{Q}\{X; 2\}/B$, the free ring on two generators modulo the relation $X_1X_2 - X_2X_1 = 1$. One verifies easily that R is simple and is a domain. However, R is left and right Goldie, as can be seen without much difficulty. (Indeed, given $f(X_1)$ and $g(X_1, X_2)$, one can show, by induction on the degree of X_2 in g, that f and g have a common left multiple. Hence the set of polynomials in X_1 is an Ore set of R and may be formally inverted to form a ring R_1. But R_1 is a principal left ideal domain and is therefore left Goldie; it follows that R is left Goldie.)

Since R is a left Goldie domain, X_1 is R-pivotal. On the other hand, if R satisfied an absolutely pivotal monomial, then R would be a simple artinian domain, i.e. a division ring, which is obviously false. Thus we have a counter-example to the question raised earlier in this section. Moreover, soc $R = 0$, by Lemma 14. Thus, a left order in a simple artinian ring may have socle 0, which is surprising in light of Theorems 6 and 11.

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